Universality in $1+1$-dimensional models

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Object of study

Processes $h(x,t)$ with following properties

1. Space $x \in \mathbf{R}$ (or maybe **Z**), time $t \in \mathbf{R}$ (or $\mathbf{R}_+$).
2. Translation invariant (modulo shifts) both in space and time.
3. Approximately local specifications only depending on 'shape'.

Question: When can one find exponents α, β and constants C_ε such that

$$H(x,t) = \lim_{\varepsilon \to 0} \varepsilon^\alpha h(x/\varepsilon, t/\varepsilon^\beta) - C_\varepsilon t$$

exists and what are possible exponents and limits?

Universality class: basin of attraction of given limit H under operation of rescaling.

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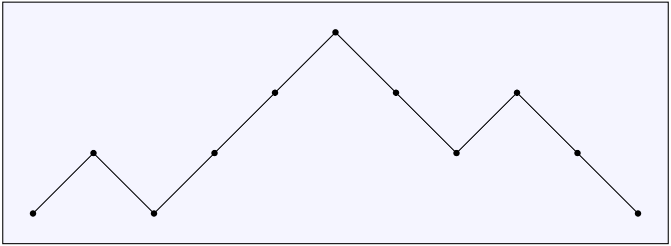
Example 1

Solid-on-solid model (also SSEP): $h\colon \mathbf{Z} \times \mathbf{R} \to \mathbf{Z}$



Theorem: Converges with $\alpha = 1/2$, $\beta = 2$ to Edwards-Wilkinson:

$$\partial_t H = \partial_x^2 H + \xi \;.$$

Symmetries: $x \leftrightarrow -x$, $t \leftrightarrow -t$, $H \leftrightarrow -H$.

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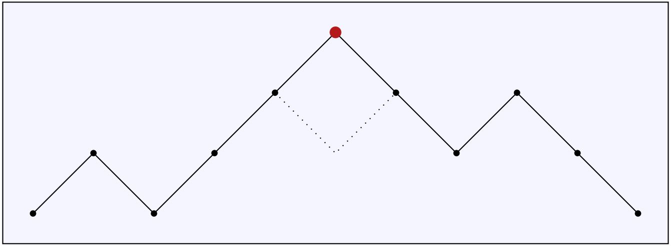

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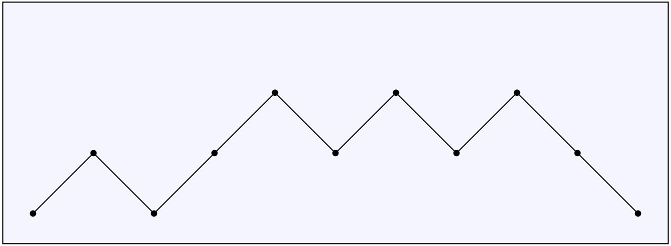

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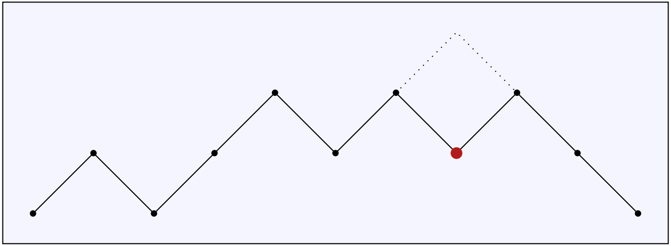

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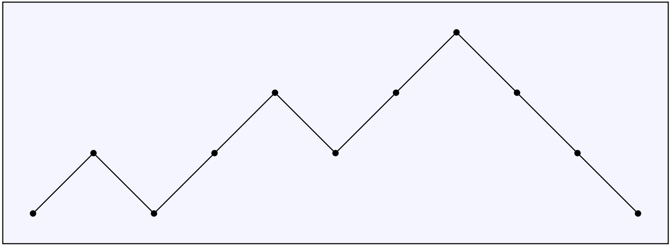

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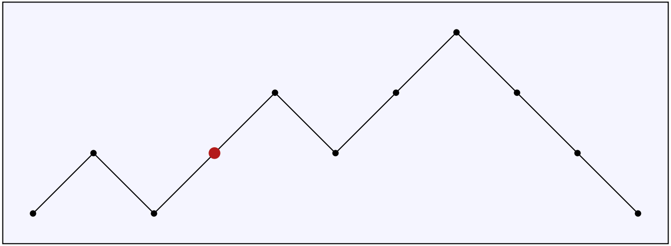

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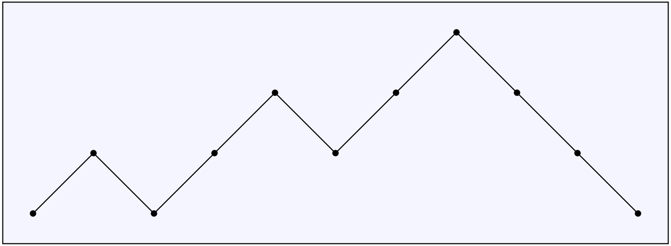

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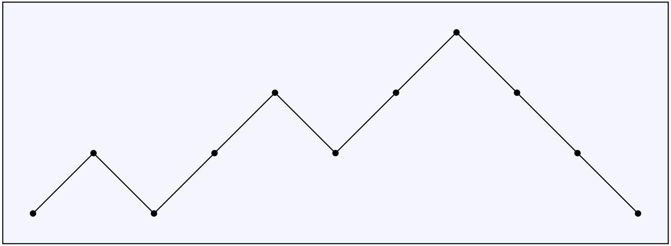

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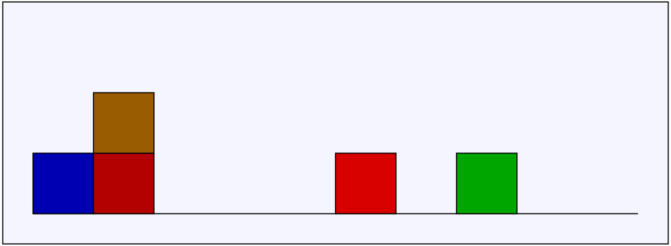

Conjecture: Converges with $\alpha = 1/2$, $\beta = 3/2$, limit: "KPZ fixed point". Surprisingly hard to show: characterisation of conjectured limit only last year! Symmetries: $x \leftrightarrow -x$, ($t \leftrightarrow -t$ & $H \leftrightarrow -H$).

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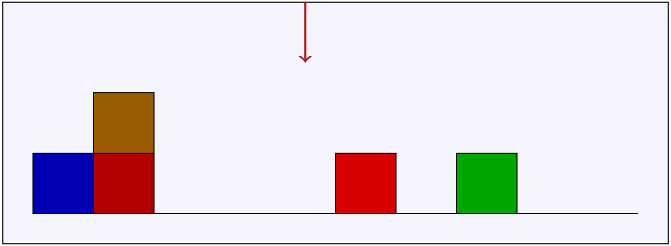

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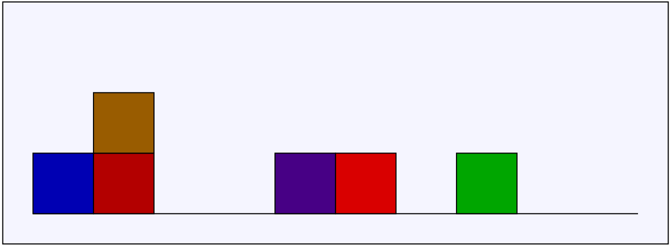

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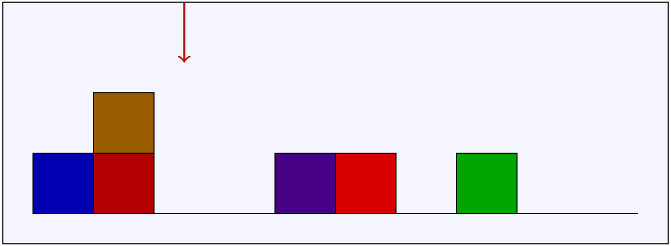

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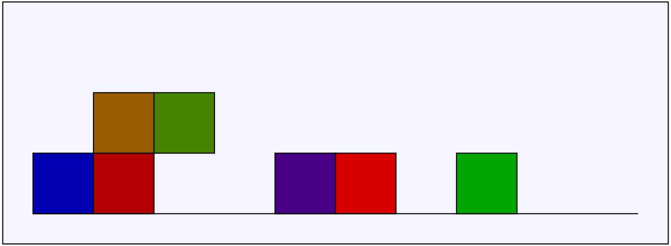

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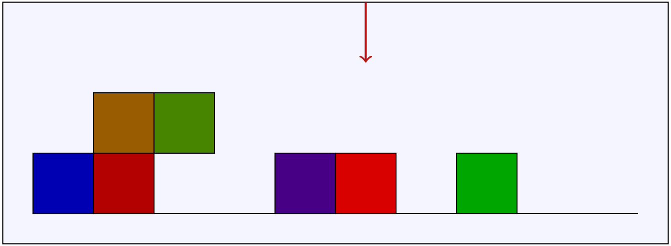

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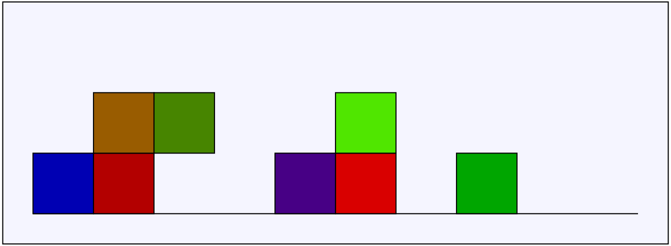

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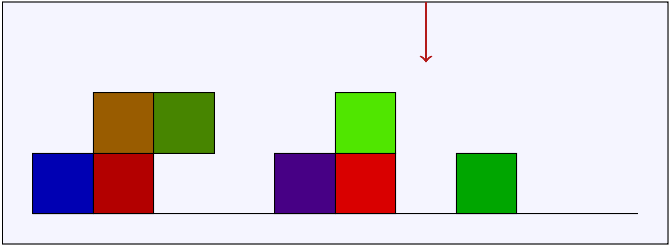

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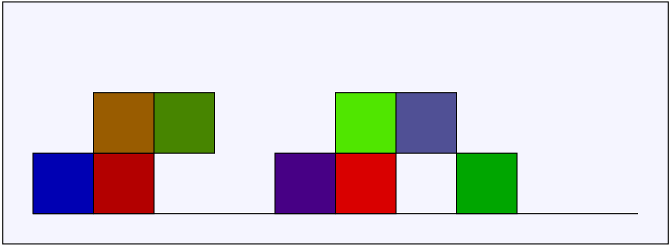

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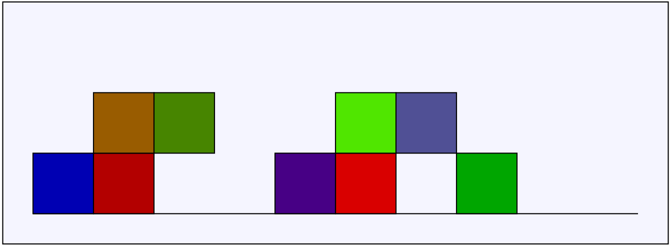

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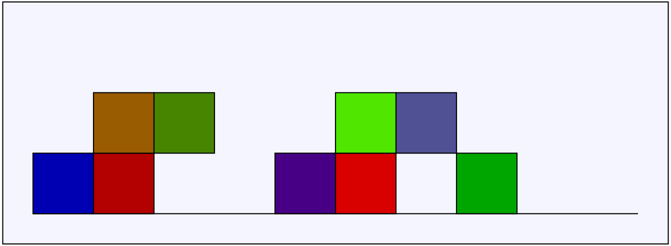

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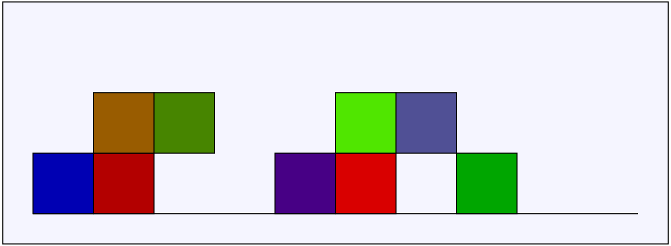

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Common properties of limits

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3. Scale invariance:

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(Relatively easy to check in both examples.) Imposes very strong constraints!

Time only: Stable Lévy processes are only possible limits.

Space-time: No known classification in general. (Exception: $2D$ conformal field theories.)

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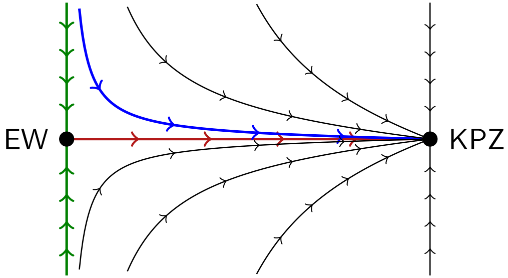

Can we understand red line and blue region? Weaker form of universality.

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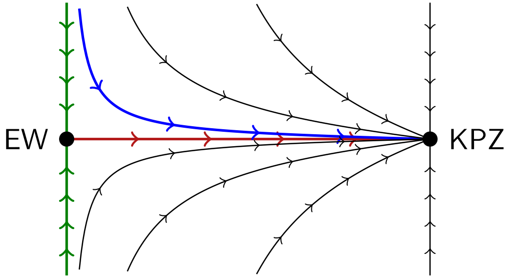

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Weak universality conjecture

Consider family of models $\alpha \mapsto h_\alpha$ such that h_0 is in EW class and h_α is in KPZ class for $\alpha \neq 0$.

Conjecture: If α is 'correctly' parametrised, one can find C_ε such that

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A natural family

Ballistic deposition can be written as

$$h(t_+, x) = \max\{h(t, x-1), h(t, x+1), h(t, x) + 1\} \ .$$

Idea: Introduce 'inverse temperature' $\beta > 0$ and consider

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$$\mathbf{P}(h_\beta(t_+, x) = y) \propto \exp(\beta y) \ .$$

The process h_∞ is BD. What is h_0 and what does it rescale to?

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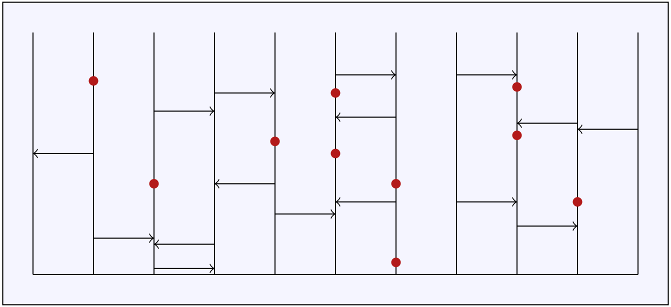
Nice graphical construction of h_0:



Scaling limit: coalescing (backwards) Brownian motions with superimposed Brownian fluctuations. Exponents $\alpha = 1$, $\beta = 2$.

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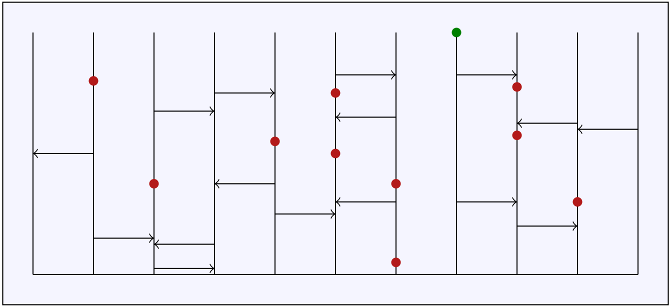

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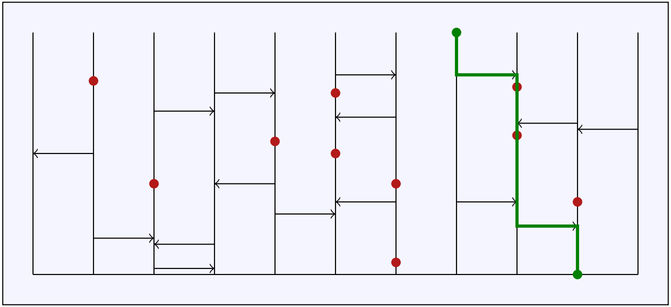

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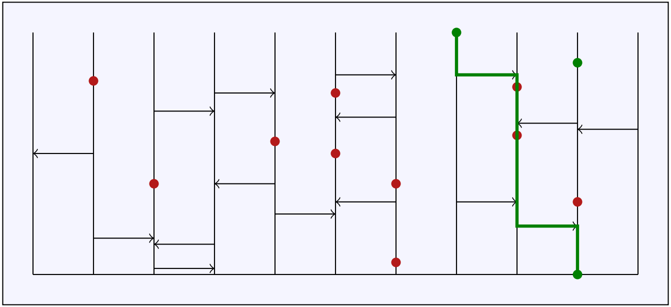

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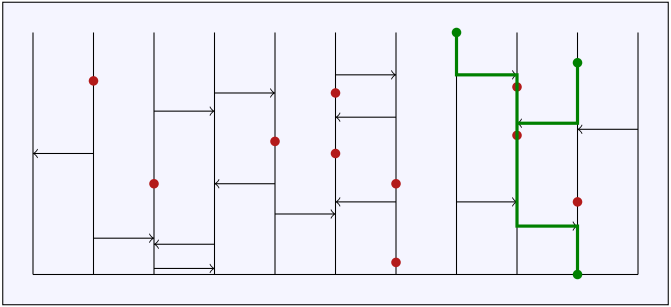

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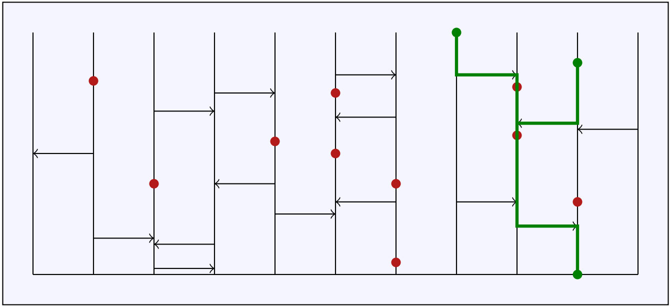

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Some properties of H

For fixed t, one has

$$\lim_{y \to x} \frac{H(t,y) - H(t,x)}{y - x} \overset{\text{law}}{=} \text{Cauchy}(1) \; .$$

In stationary case $H(t,y) - H(t,x) \overset{\text{law}}{=} \text{Cauchy}(y - x)$.

Question: Is $H(t, \cdot)$ a Cauchy process? No.

Theorem: The law of $(H(t,x))_{x \in [0,1]}$ is singular with respect to the law of the Cauchy process.

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Link with Brownian web

Dual web: shocks.

7 types of possible points in Brownian web:

All play a role in the lifespan of a singularity and its basin of attraction!

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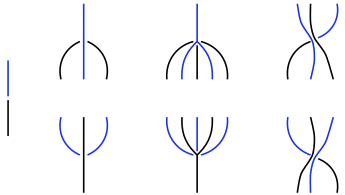

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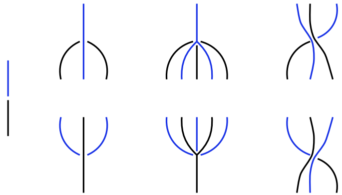

All play a role in the lifespan of a singularity and its basin of attraction!

The end

Thank you for

your attention!