
Coarse-graining of collective dynamics models

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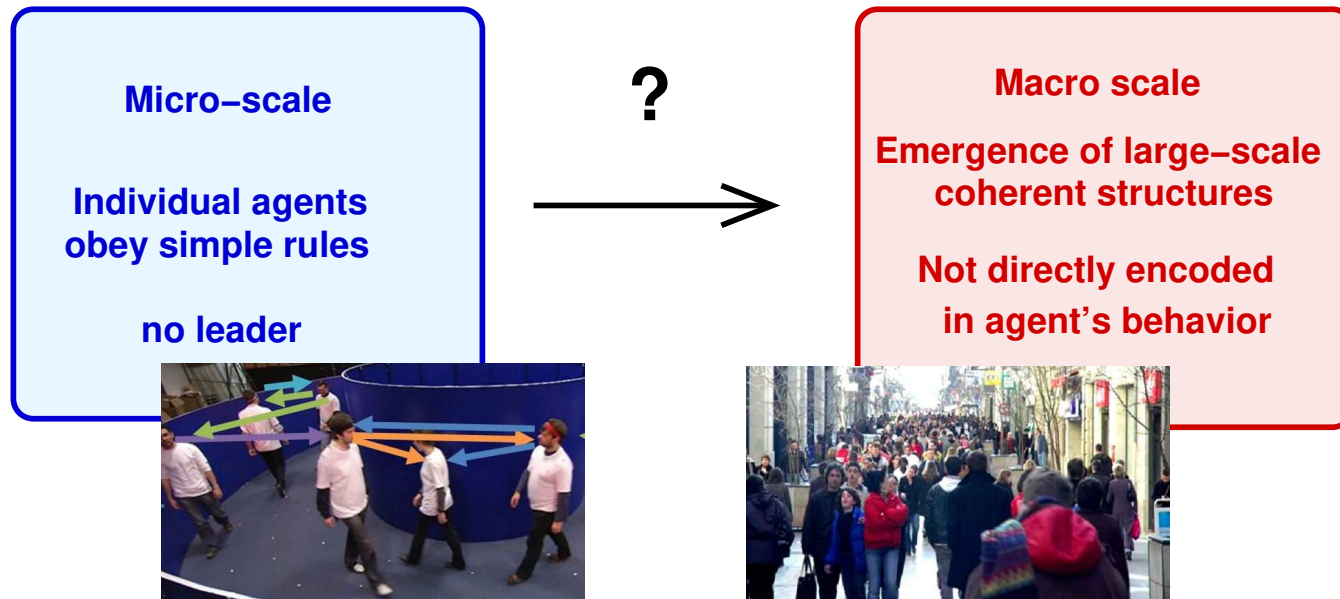
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1. Individual-based model
2. Mean-Field model
3. Self-Organized Quaternionic Hydrodynamics (SOHQ)
4. Comparison with SOH dynamics for Vicsek
5. Conclusion

1. Individual-Based Model



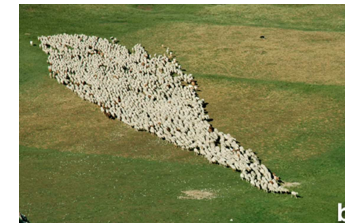
Link micro to macro scales

Lack of conservations
Breakdown of chaos property



Phase transitions

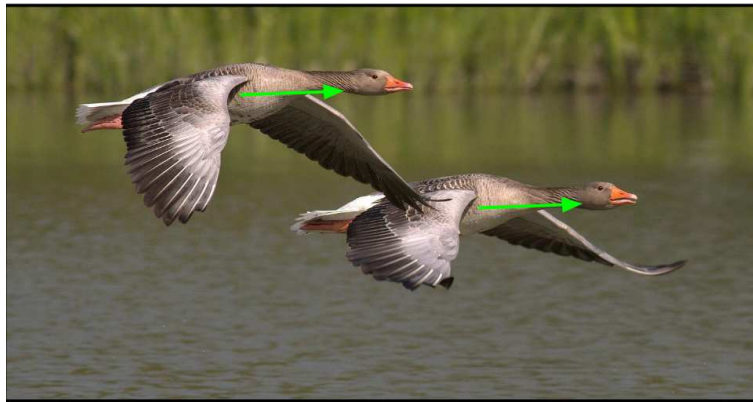
Symmetry-breaking
Jamming
Continuum to network



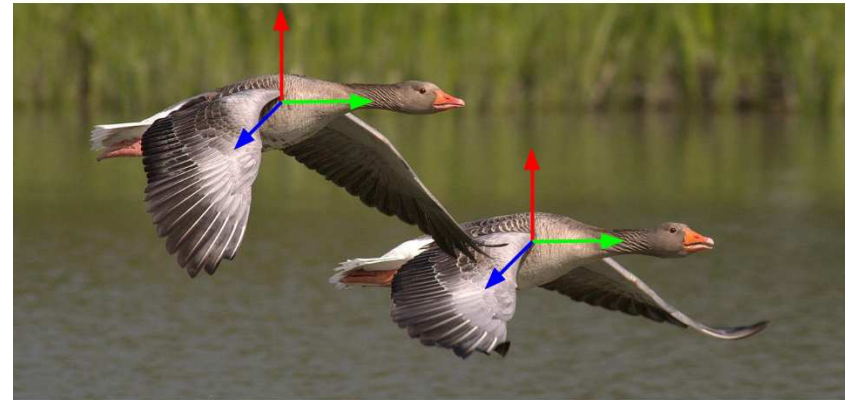
Self-propelled agents which align with their neighbors

Case 1: Alignment of their directions of motion (Vicsek)

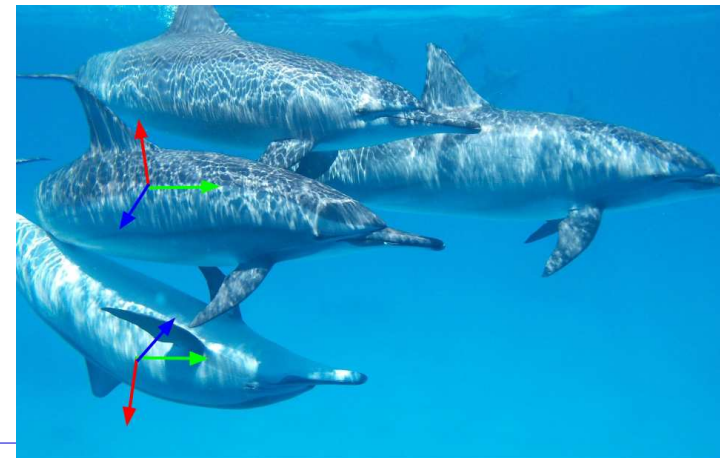
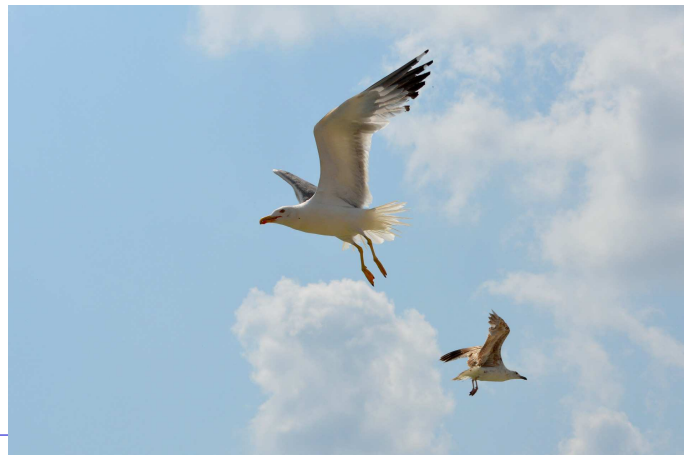
Case 2: Alignment of their full body attitude (new model)



Vicsek model



Body attitude alignment



Individual-Based (aka particle) model

self-propelled \Rightarrow all particles have same constant speed a
align with their neighbours up to a certain noise

$X_k(t) \in \mathbb{R}^d$: position of the k -th particle at time t

$V_k(t) \in \mathbb{S}^{d-1}$: velocity orientation ($|V_k(t)| = 1$)

$$\dot{X}_k(t) = aV_k(t)$$

$$dV_k(t) = P_{V_k^\perp} \circ (\nu \bar{V}_k dt + \sqrt{2\tau} dB_t^k), \quad P_{V_k^\perp} = \text{Id} - V_k \otimes V_k$$

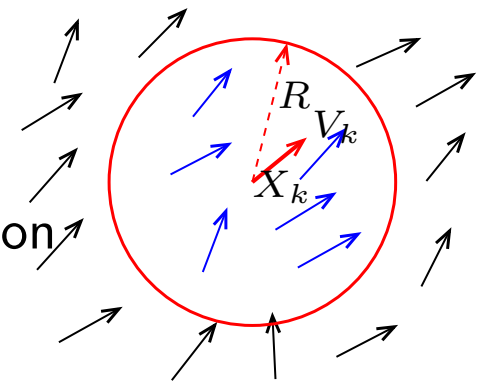
$$\mathcal{J}_k = \sum_{j, |X_j - X_k| \leq R} V_j, \quad \bar{V}_k = \frac{\mathcal{J}_k}{|\mathcal{J}_k|}$$

ν alignment frequency; τ noise intensity

\mathcal{J}_k, \bar{V}_k neighbors' mean velocity, mean orientation

$P_{V_k^\perp}$ projection on V_k^\perp , maintains $|V_k(t)| = 1$

\circ indicates Stratonovich SDE



$X_k(t) \in \mathbb{R}^d$: position of the k -th subject at time t

$A_k(t) \in \text{SO}(d)$: rotation mapping reference frame (e_1, \dots, e_d) to subject's body frame

$A_k(t)e_1 \in \mathbb{S}^{d-1}$: propulsion direction

$$\dot{X}_k(t) = aA_k(t)e_1$$

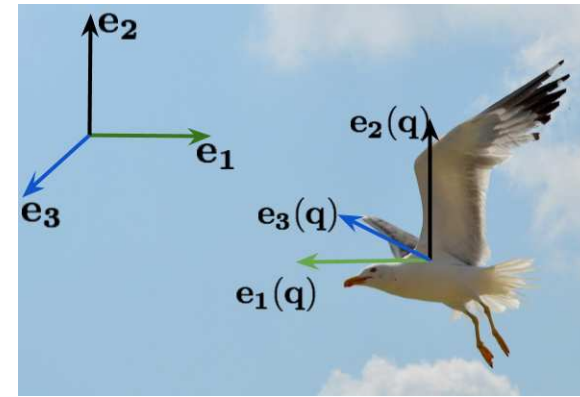
$$dA_k(t) = P_{T_{A_k(t)}\text{SO}(d)} \circ (\nu \bar{A}_k dt + \sqrt{2\tau} dB_t^k),$$

$$M_k(t) = \sum_{j, |X_j - X_k| \leq R} A_j(t), \quad \bar{A}_k = \text{PD}(M_k(t))$$

M_k arithmetic mean of neighbors' A matrices

$A = \text{PD}(M) \Leftrightarrow \exists S$ symmetric s.t. $M = AS$ (polar decomp.)

$P_{T_{A_k(t)}\text{SO}(d)}$ projection on the tangent $T_{A_k(t)}\text{SO}(d)$, maintains $A_k(t) \in \text{SO}(d)$



Can we quantify the difference between the two models ?

Is body-alignment just Vicsek for direction of motion
with frame dynamic superimposed to it ?

Or does body-alignment provide genuinely new dynamic ?
i.e. do gradients of body frames orientation influence
direction of motion ?

Not easy to answer with Individual-Based Model

Goal: use coarse-grained model to answer this question

Quaternions: $q = q_0 + q_1i + q_2j + q_3k$, $q_0, \dots, q_3 \in \mathbb{R}$.

$i^2 = j^2 = k^2 = ijk = -1$: division ring \mathbb{H} (non commutative)

$q = \operatorname{Re}q + \operatorname{Im}q$ with $\operatorname{Re}q = q_0$, $\operatorname{Im}q = q_1i + q_2j + q_3k$

$\mathbb{R}^3 \ni \vec{q} = (q_1, q_2, q_3) \approx q = q_1i + q_2j + q_3k \in \{q \in \mathbb{H}, \operatorname{Re}q = 0\}$

Conjugate $q^* = \operatorname{Re}q - \operatorname{Im}q$

Scalar product $p \cdot q = pq^* = \operatorname{Re}p \operatorname{Re}q + \operatorname{Im}p \cdot \operatorname{Im}q$

Unitary quaternions $\mathbb{H}_1 = \{q \in \mathbb{H}, qq^* = 1\} \approx \mathbb{S}^3$

$\mathbb{H}_1 \ni q = \cos(\theta/2) + \sin(\theta/2)n$, $\theta \in [0, 2\pi)$, $\vec{n} \in \mathbb{S}^2$

The map $\mathbb{R}^3 \ni \vec{v} \rightarrow \operatorname{Im}(qvq^*) \in \mathbb{R}^3$ is rotation axis n angle θ

Given $A \in \operatorname{SO}(3)$ encoded by q and $-q \in \mathbb{H}_1$

$A(q_1)A(q_2) = A(q_1q_2)$

$X_k(t) \in \mathbb{R}^d$: position of the k -th subject at time t

$q_k(t) \in \mathbb{H}_1$: quaternion encoding rotation mapping reference frame
 $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to subject's body frame

$\vec{e}_1(q_k(t)) = \text{Im}(q_k(t) e_1 q_k(t)^*) \in \mathbb{S}^{d-1}$: propulsion direction

$$\dot{X}_k(t) = a\vec{e}_1(q_k(t))$$

$$dq_k(t) = P_{q_k(t)^\perp} \circ (\nu F_k(t)dt + \sqrt{\tau/2} dB_t^k),$$

$$F_k(t) = (\bar{q}_k(t) \cdot q_k(t)) \bar{q}_k(t)$$

$\bar{q}_k(t)$ leading eigenvector of tensor

$$Q_k(t) = \sum_{j, |X_j - X_k| \leq R} q_j(t) \otimes q_j(t)$$

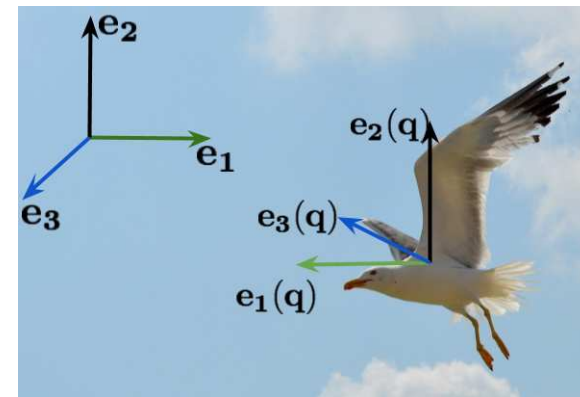
$Q_k(t)$ de Gennes Q-tensor; $\bar{q}_k(t)$ mean nematic alignment direction

Describes alignment of q_k with \bar{q}_k or $-\bar{q}_k$

$P_{q_k(t)^\perp}$ projection on q_k^\perp , maintains $q_k q_k^* = 1$

Similarity with polymer models

Quaternion dynamics identical to previous rotation matrix dynamics



2. Mean-Field model

$f(x, q, t)$ = particle probability density $x \in \mathbb{R}^3$, $q \in \mathbb{H}_1$
 satisfies a Fokker-Planck equation

$$\partial_t f + a\vec{e}_1(q) \cdot \nabla_x f + \nabla_q \cdot (\mathcal{F}_f f) = (\tau/4)\Delta_q f$$

$$\mathcal{F}_f(x, q, t) = \nu P_{q^\perp} ((\bar{q}_f(x, t) \cdot q)\bar{q}_f(x, t)), \quad P_{q^\perp} = \text{Id} - q \otimes q$$

$\bar{q}_f(x, t)$ = leading eigenvector of tensor

$$\mathcal{Q}_f(x, t) = \int_{|x'-x| < R} \int_{\mathbb{H}_1} f(x', q', t) (q' \otimes q') dq' dx'$$

$\mathcal{Q}_f(x, t)$ = Q-tensor in a neighborhood of x

$(\bar{q}_f(x, t) \cdot q)\bar{q}_f(x, t)$ provides nematic alignment of q with $\bar{q}_f(x, t)$

$\mathcal{F}_f(x, q, t)$ = projection of nematic alignment direction on q^\perp

$(x, q) \in \mathbb{R}^3 \times \mathbb{H}_1$; $\nabla_{q^\cdot}, \nabla_q$: div and grad on \mathbb{H}_1

Δ_q Laplace-Beltrami operator on $\mathbb{H}_1 \approx \mathbb{S}^3$

Highlights important **physical scales & small parameters**

Choose time scale t_0 , space scale $x_0 = at_0$

Set f scale $f_0 = 1/x_0^3$, F scale $\mathcal{F}_0 = 1/t_0$

Introduce **dimensionless parameters** $\bar{\nu} = \nu t_0$, $\bar{\tau} = \tau t_0$, $\bar{R} = \frac{R}{x_0}$

Change variables $x = x_0 x'$, $t = t_0 t'$, $f = f_0 f'$, $\mathcal{F} = \mathcal{F}_0 \mathcal{F}'$

Get the **scaled** Fokker-Planck system (omitting the primes):

$$\partial_t f + \vec{e}_1(q) \cdot \nabla_x f + \nabla_q \cdot (\mathcal{F}_f f) = (\bar{\tau}/4) \Delta_q f$$

$$\mathcal{F}_f(x, q, t) = \bar{\nu} P_{q^\perp} \left((\bar{q}_f(x, t) \cdot q) \bar{q}_f(x, t) \right), \quad P_{q^\perp} = \text{Id} - q \otimes q$$

$\bar{q}_f(x, t) =$ leading eigenvector of tensor

$$\mathcal{Q}_f(x, t) = \int_{|x'-x| < \bar{R}} \int_{\mathbb{H}_1} f(x', q', t) (q' \otimes q') dq' dx'$$

Choice of t_0 such that $\bar{\tau} = \frac{1}{\varepsilon}, \varepsilon \ll 1$

Macroscopic scale:

there are many velocity diffusion events within one time unit

Assumption 1: $k := \frac{\bar{\nu}}{\bar{\tau}} = \mathcal{O}(1)$

Social interaction and diffusion act at the same scale

Implies $\bar{\nu}^{-1} = \mathcal{O}(\varepsilon)$, i.e. mean-free path is microscopic

Assumption 2: $\bar{R} = \varepsilon$

Interaction range is microscopic

and of the same order as mean-free path $\bar{\nu}^{-1}$

Possible variant: $\bar{R} = \mathcal{O}(\sqrt{\varepsilon})$: interaction range still small

but large compared to mean-free path. To be investigated later

With Assumption 2 ($\bar{R} = \mathcal{O}(\varepsilon)$)

Interaction is local at leading order: by Taylor expansion:

$$Q_f = Q_f + \mathcal{O}(\varepsilon^2), \quad Q_f(x, t) = \int_{\mathbb{H}_1} f(x, q', t) (q' \otimes q') dq'$$

$Q_f(x, t) =$ local Q-tensor. From now on, **neglect $\mathcal{O}(\varepsilon^2)$ term**

Fokker-Planck eq. in scaled variables

$$\varepsilon (\partial_t f^\varepsilon + \vec{e}_1(q) \cdot \nabla_x f^\varepsilon) = -\nabla_q \cdot (F_{f^\varepsilon} f^\varepsilon) + \Delta_q f^\varepsilon$$

$$F_f(x, q, t) = 4k P_{q^\perp} ((\bar{q}_f(x, t) \cdot q) \bar{q}_f(x, t)), \quad P_{q^\perp} = \text{Id} - q \otimes q$$

$\bar{q}_f(x, t) =$ leading eigenvector of tensor

$$Q_f(x, t) = \int_{\mathbb{H}_1} f(x, q', t) (q' \otimes q') dq'$$

Coarse-grained model is obtained in the **limit $\varepsilon \rightarrow 0$**

3. Self-Organized Quaternionic Hydrodynamics (SOHQ)

Model can be written

$$\partial_t f^\varepsilon + e_1(q) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} C(f^\varepsilon)$$

with **collision operator**

$$C(f) = -\nabla_q \cdot (F_f f) + \Delta_q f$$

$$F_f = 4k P_{q^\perp} ((\bar{q}_f \cdot q) \bar{q}_f)$$

\bar{q}_f leading eigenvector of Q_f

$$Q_f = \int_{\mathbb{H}_1} f(q') (q' \otimes q') dq'$$

When $\varepsilon \rightarrow 0$, $f^\varepsilon \rightarrow f$ (formally) such that $C(f) = 0$

\Rightarrow importance of the solutions of **$C(f) = 0$ (equilibria)**

C acts on q -variable only ((x, t) are just parameters)

Force F_f can be written: $F_f(v) = 2k \nabla_q ((\bar{q}_f \cdot q)^2)$

Note \bar{q}_f independent of q ((x, t) are fixed)

Rewrite:

$$\begin{aligned} C(f)(q) &= \nabla_q \cdot \left[-2k f \nabla_q ((\bar{q}_f \cdot q)^2) + \nabla_q f \right] \\ &= \nabla_q \cdot \left[f \nabla_q (-2k (\bar{q}_f \cdot q)^2 + \ln f) \right] \end{aligned}$$

Let $\bar{q} \in \mathbb{H}_1$ be given: Solutions of

$\nabla_q (-2k (\bar{q}_f \cdot q)^2 + \ln f) = 0$ are proportional to :

$$f(v) = M_{\bar{q}}(q) := \frac{1}{Z} \exp (2k(\bar{q} \cdot q)^2) \text{ with } \int_{\mathbb{H}_1} M_{\bar{q}}(q) dq = 1$$

'generalized' von Mises-Fisher (VMF) distribution

Again:

$$M_{\bar{q}}(q) := \frac{e^{2k(q \cdot \bar{q})^2}}{\int_{\mathbb{H}_1} e^{2k(q' \cdot \bar{q})^2} dq'}$$

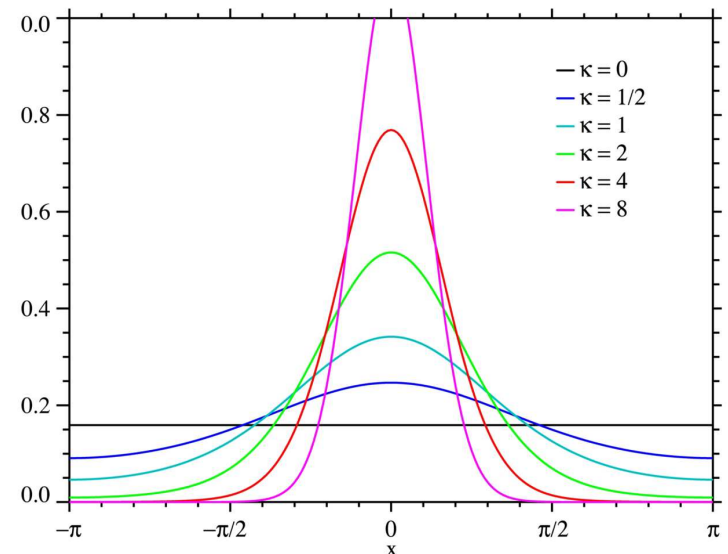
$k > 0$: concentration parameter; $\bar{q} \in \mathbb{H}_1 \approx \mathbb{S}^3$: orientation

Order parameter: $c_1(k)$ s.t. $\int_{\mathbb{H}_1} M_{\bar{q}}(q) e_1(q) dq = c_1(k) e_1(\bar{q})$

$$k \nearrow c_1(k), \quad 0 \leq c_1(k) \leq 1$$

Here:

concentration parameter k
and order parameter $c_1(k)$
are constant



Definition: equilibrium manifold $\mathcal{E} = \{f(q) \mid C(f) = 0\}$

Theorem: $\mathcal{E} = \{\rho M_{\bar{q}} \text{ for arbitrary } \rho \in \mathbb{R}_+ \text{ and } \bar{q} \in \mathbb{H}_1\}$

Note: $\dim \mathcal{E} = 4$

Proof: follows from entropy inequality:

$$H(f) = \int C(f) \frac{f}{M_{\bar{q}_f}} dq = - \int M_{\bar{q}_f} \left| \nabla_q \left(\frac{f}{M_{\bar{q}_f}} \right) \right|^2 \leq 0$$

$$\text{follows from } C(f) = \nabla_q \cdot [M_{\bar{q}_f} \nabla_q \left(\frac{f}{M_{\bar{q}_f}} \right)]$$

Then, $C(f) = 0$ implies $H(f) = 0$ and $\frac{f}{M_{\bar{q}_f}} = \text{Constant}$
and f is of the form $\rho M_{\bar{q}}$

Reciprocally, if $f = \rho M_{\bar{q}}$, then, $\bar{q}_f = \bar{q}$ and $C(f) = 0$

$f^\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$ with $q \rightarrow f(x, q, t) \in \mathcal{E}$ for all (x, t)

Implies that $f(x, q, t) = \rho(x, t)M_{\bar{q}(x, t)}(q)$

Need to specify the dependence of ρ and \bar{q} on (x, t)

Requires 4 equations since $(\rho, \bar{q}) \in \mathbb{R}_+ \times \mathbb{H}_1 \approx \mathbb{R}_+ \times \mathbb{S}^3$ are determined by 4 independent real quantities

f satisfies

$$\partial_t f + e_1(q) \cdot \nabla_x f = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} C(f^\varepsilon)$$

Problem: $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} C(f^\varepsilon)$ is not known

Trick:

Collision invariant

is a function $\psi(q)$ such that $\int C(f)\psi dq = 0, \quad \forall f$

Form a linear vector space \mathcal{CI}

Multiply eq. by ψ : ε^{-1} term disappears

Find a conservation law:

$$\partial_t \left(\int_{\mathbb{H}_1} f(x, q, t) \psi(q) dq \right) + \nabla_x \cdot \left(\int_{\mathbb{H}_1} f(x, q, t) \psi(q) e_1(q) dq \right) = 0$$

Have used that ∂_t or ∇_x and $\int \dots dq$ can be interchanged

Limit fully determined if $\dim \mathcal{CI} = \dim \mathcal{E} = 4$

$\mathcal{CI} = \text{Span}\{1\}$. Interaction preserves mass but no other quantity

Due to self-propulsion, **no momentum conservation**

$\dim \mathcal{CI} = 1 < \dim \mathcal{E} = 4$. Is the limit problem ill-posed ?

Proof that $\psi(q) = 1$ is a CI ?

Obvious. $C(f) = \nabla_q \cdot [\dots]$ is a divergence

By Stokes theorem on the sphere, $\int C(f) dq = 0$

Use of the CI $\psi(q) = 1$: Get the **conservation law**

$$\partial_t \left(\int_{\mathbb{H}_1} f(x, q, t) dq \right) + \nabla_x \cdot \left(\int_{\mathbb{H}_1} f(x, q, t) e_1(q) dq \right) = 0$$

With $f = \rho M_{\bar{q}}$ we have

$$\int f(x, v, t) dv = \rho(x, t), \quad \int f(x, v, t) e_1(q) dq = c_1 \rho(x, t) e_1(\bar{q}(x, t))$$

We end up with the **mass conservation eq.**

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho e_1(\bar{q})) = 0$$

Given $\bar{q}_0 \in \mathbb{H}_1$, Define $\mathcal{R}_{\bar{q}_0}(f) = \nabla_q \cdot [M_{\bar{q}_0} \nabla_q (\frac{f}{M_{\bar{q}_0}})]$

Note $f \rightarrow \mathcal{R}_{\bar{q}_0}(f)$ is linear and $C(f) = \mathcal{R}_{\bar{q}_f}(f)$

A function $\psi_{\bar{q}_0}(q)$ is a **GCI associated to \bar{q}_0** , iff

$$\int \mathcal{R}_{\bar{q}_0}(f) \psi_{\bar{q}_0} dq = 0, \quad \forall f \text{ such that } P_{q_0^\perp} \left[\left(\int_{\mathbb{H}_1} f(q) (q \otimes q) dq \right) \bar{q}_0 \right] = 0$$

The set of GCI $\mathcal{G}_{\bar{q}_0}$ is a linear vector space

Theorem: Given $\bar{q}_0 \in \mathbb{H}_1$, $\mathcal{G}_{\bar{q}_0}$ is the 4-dim vector space :

$$\mathcal{G}_{\bar{q}_0} = \{q \mapsto \alpha + h(q \cdot \bar{q}_0) \beta \cdot q, \text{ with arbitrary } \alpha \in \mathbb{R} \text{ and } \beta \in \mathbb{H} \text{ with } \beta \cdot \bar{q}_0 = 0\}.$$

Introduce $r = q \cdot \bar{q}_0 \in [-1, 1]$. h is the unique solution in V of:

$$-(1-r^2)^{-3/2} \exp(-2kr^2) \frac{d}{dr} \left[(1-r^2)^{5/2} \exp(2kr^2) \frac{dh}{dr} \right] + (4kr^2 + 3) h(r) = -r$$

$$V = \{h \mid (1-r^2)^{3/4} h \in L^2(-1, 1), (1-r^2)^{5/4} h' \in L^2(-1, 1)\}$$

Furthermore, h is odd and non-positive for $r \geq 0$

Use GCI $h(q \cdot \bar{q}_0)\beta \cdot q$ for $\beta \in \mathbb{H}$ with $\beta \cdot \bar{q}_0 = 0$

Equivalently, use the quaternion valued function

$$\psi_{\bar{q}_0}(q) = h(q \cdot \bar{q}_0)P_{\bar{q}_0^\perp} q$$

Multiply FP eq by GCI $\psi_{\bar{q}_{f^\varepsilon}}$: $O(\varepsilon^{-1})$ terms disappear

$$\int C(f) \vec{\psi}_{\bar{q}_f} dv = \int \mathcal{R}_{\bar{q}_f}(f) \psi_{\bar{q}_f} dq = 0 \quad \text{by property of GCI}$$

Gives:
$$\int (\partial_t f^\varepsilon + e_1(q) \cdot \nabla_x f^\varepsilon) \psi_{\bar{q}_{f^\varepsilon}} dq = 0$$

As $\varepsilon \rightarrow 0$: $f^\varepsilon \rightarrow \rho M_{\bar{q}}$ and $\psi_{\bar{q}_{f^\varepsilon}} \rightarrow \psi_{\bar{q}}$ Leads to:

$$\int (\partial_t(\rho M_{\bar{q}}) + e_1(q) \cdot \nabla_x(\rho M_{\bar{q}})) \psi_{\bar{q}} dq = 0$$

Not a conservation equation

because of dependence of $\psi_{\bar{q}}$ upon (x, t) through \bar{q}

∂_t or ∇_x and $\int \dots dq$ cannot be interchanged

Takes the form:

$$\begin{aligned} \rho \left(\partial_t \bar{q} + c_2 (e_1(\bar{q}) \cdot \nabla_x) \bar{q} \right) + c_3 [e_1(\bar{q}) \times \nabla_x \rho] \bar{q} \\ + c_4 \rho [(\nabla_{x,\text{rel}} \bar{q}) e_1(\bar{q}) + (\nabla_{x,\text{rel}} \cdot \bar{q}) e_1(\bar{q})] \bar{q} = 0 \end{aligned}$$

where

$$\begin{aligned} (\nabla_{x,\text{rel}} \bar{q}) &= (\partial_{x_i,\text{rel}} \bar{q})_{i=1,2,3} = ((\partial_{x_i} \bar{q}) \bar{q}^*)_{i=1,2,3} \in \mathbb{H}_{\text{Im}}^3 \\ (\nabla_{x,\text{rel}} \cdot \bar{q}) &= \sum_{i=1,2,3} (\partial_{x_i,\text{rel}} \bar{q})_i = \sum_{i=1,2,3} ((\partial_{x_i} \bar{q}) \bar{q}^*)_i \in \mathbb{R} \end{aligned}$$

$$\mathbb{H}_{\text{Im}} = \{q \in \mathbb{H}, \text{Re}q = 0\} \approx \mathbb{R}^3$$

$$(\partial_{x_i,\text{rel}} \bar{q})_j = j\text{-th component of } \partial_{x_i,\text{rel}} \bar{q}$$

$$(\nabla_{x,\text{rel}} \bar{q}) e_1(\bar{q}) = ((\partial_{x_i,\text{rel}} \bar{q}) \cdot e_1(\bar{q}))_{i=1,2,3}$$

Coefficients c_2 and c_4 depend on GCI h

Self-Organized Quaternionic Hydrodynamics (SOHQ)

System for density $\rho(x, t)$ and quaternion orientation $\bar{q}(x, t)$:

$$\partial_t \rho + c_1 \nabla_x (\rho e_1(\bar{q})) = 0$$

$$\begin{aligned} \rho \left(\partial_t \bar{q} + c_2 (e_1(\bar{q}) \cdot \nabla_x) \bar{q} \right) + c_3 [e_1(\bar{q}) \times \nabla_x \rho] \bar{q} \\ + c_4 \rho \left[(\nabla_{x,\text{rel}} \bar{q}) e_1(\bar{q}) + (\nabla_{x,\text{rel}} \cdot \bar{q}) e_1(\bar{q}) \right] \bar{q} = 0 \end{aligned}$$

$$|\bar{q}| = 1$$

4. Comparison with SOH dynamics for Vicsek

Vicsek mean-field model for $f^\varepsilon(x, v, t)$

position $x \in \mathbb{R}^3$, velocity orientation $v \in \mathbb{S}^2$

As $\varepsilon \rightarrow 0$, $f^\varepsilon(x, v, t) \rightarrow \rho(x, t) \mathcal{M}_{\Omega(x, t)}(v)$

$\rho(x, t) \geq 0$, $\Omega(x, t) \in \mathbb{S}^2$

$\mathcal{M}_\Omega(v) = \frac{1}{Z} \exp(k(\Omega \cdot v))$, $\int_{\mathbb{S}^2} \mathcal{M}_\Omega(v) dv = 1$

$(\rho(x, t), \Omega(x, t))$ solves SOH model:

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho (\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega) + c_3 P_{\Omega^\perp} \nabla_x \rho = 0, \quad P_{\Omega^\perp} = \text{Id} - \Omega \otimes \Omega$$

$$|\Omega| = 1$$

Similar to **Compressible Euler eqs.** of gas dynamics

System of **hyperbolic** eqs.

But **major differences**:

Geometric constraint $|\Omega| = 1$: requires P_{Ω^\perp} to be maintained

System is **non conservative** due to the presence of P_{Ω^\perp}

$c_2 \neq c_1$: **loss of Galilean invariance**

$\bar{q}(x, t) \in \mathbb{H}_1$ encodes rotation $\Lambda(x, t) \in SO(3)$

$\Lambda(x, t)$ describes agents' local average body attitude

$\Omega(x, t) = \Lambda(x, t)e_1 = e_1(\bar{q}(x, t))$: direction of motion

$u(x, t) = \Lambda(x, t)e_2 = e_2(\bar{q}(x, t))$: belly to back

$v(x, t) = \Lambda(x, t)e_3 = e_3(\bar{q}(x, t))$: right to left wing

SOHQ model equivalent to

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho (\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega) + P_{\Omega^\perp} (c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v)) = 0$$

$$\rho (\partial_t u + c_2 (\Omega \cdot \nabla_x) u) - u \cdot (c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v)) \Omega + c_4 \rho \delta(\Omega, u, v) v = 0$$

$$\rho (\partial_t v + c_2 (\Omega \cdot \nabla_x) v) - v \cdot (c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v)) \Omega - c_4 \rho \delta(\Omega, u, v) u = 0$$

$$|\Omega| = |u| = |v| = 1, \quad \Omega \cdot u = u \cdot v = v \cdot \Omega = 0$$

with $r(\Omega, u, v)$ (for rotational) and $\delta(\Omega, u, v)$ (for divergence):

$$r(\Omega, u, v) = (\Omega \cdot \nabla_x) \Omega + (u \cdot \nabla_x) u + (v \cdot \nabla_x) v \in \mathbb{R}^3$$

$$\delta(\Omega, u, v) u = [(\Omega \cdot \nabla_x) u] \cdot v + [(u \cdot \nabla_x) v] \cdot \Omega + [(v \cdot \nabla_x) \Omega] \cdot u \in \mathbb{R}$$

Compare eqs for ρ and Ω :

SOH: Coarse-grained Vicsek model

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho (\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega) + c_3 P_{\Omega^\perp} \nabla_x \rho = 0$$

SOHQ: Coarse-grained body orientation model

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho (\partial_t \Omega + c_2 (\Omega \cdot \nabla_x) \Omega) + P_{\Omega^\perp} (c_3 \nabla_x \rho - c_4 \rho r(\Omega, u, v)) = 0$$

Difference is the term

$$r(\Omega, u, v) = (\Omega \cdot \nabla_x) \Omega + (u \cdot \nabla_x) u + (v \cdot \nabla_x) v$$

Shows how differences in body orientation affect direction of the flock

Can we quantify the difference between the Vicsek and body alignment models ?

YES: by using coarse-grained models SOH and SOHQ respectively

Is body-alignment just Vicsek for direction of motion with frame dynamic superimposed to it ?

Answer is 'NO'

Or does body-alignment provide genuinely new dynamic ?
i.e. do gradients of body frames orientation influence direction of motion ?

Answer is 'YES'

5. Conclusion

New collective dynamics model relying on full body alignment
body frame alignment \Leftrightarrow quaternion nematic alignment

Coarse-grained model is SOHQ

First order PDE for density and local average quaternion
describes dynamics of agents' local mean body frame
dynamics genuinely \neq from velocity alignment (Vicsek or SOH)

Perspectives

analysis of the model
rigorous proof of convergence
numerical simulations
Higher dimensions